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TECHNICAL NOTE

The analogy between fluid friction and heat transfer of laminar natural convection on vertical and horizontal flat plates

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1. INTRODUCTION

REYNOLDS [1] noted that the basic mechanisms of momentum and heat transfer are essentially the same for a Prandtl number of unity. He established a simple proportional relation between the fluid friction and heat transfer for $Pr = 1$. Since then a lot of modified analogies for large Prandtl numbers have been developed by many authors. Among them the analogies of Prandtl [2], von Karman [3], Colburn [4], and Friend and Metzner [5] are the most noteworthy. These analogies can be developed from the correlations of experimental data and/or from theoretical analyses. The analogies between the wall shear stress and the heat or mass flux are universal relations valid not only for turbulent forced convection in conduits but also for laminar and turbulent forced convection on a flat plate. Through these analogies, we may estimate the heat or mass transfer rate from the analytical, numerical, or experimental results of wall friction, and vice versa. The intimate relations between momentum, heat, and mass transfer are quite useful.

In spite of the great success of the analogies between friction and heat or mass transfer of forced convection, the analogy of natural convection has not been developed. However, Larsen and Arpaci [6] have pointed out the possibility of an approximate analogy between heat transfer and friction of laminar natural convection.

In this paper, we develop the analogies between the friction coefficient and the Nusselt number of laminar natural convection on vertical and horizontal flat plates with uniform wall temperature (UWT) or uniform wall heat flux (UHF). The analogies for large Prandtl numbers ($Pr \geqslant 0.7$) and those for any Prandtl number between 0.001 and infinity are proposed. With these analogies, one can estimate the heat transfer rate or the wall temperature from the experimental results of fluid dynamics.

2. ANALOGIES

2.1. *Formulations*

The dimensional analysis of Arpaci and Larsen [6, 7] suggests a proper dimensionless number for laminar natural convection

$$\Pi = \frac{Pr}{1+Pr} Ra \tag{1}$$

where the local Rayleigh number Ra for natural convection heat transfer is defined as

$$Ra = g\beta T^* x^3/\alpha\nu \tag{2}$$

with the characteristic temperature

$$T^* = T_0 - T_\infty \quad \text{for UWT} \tag{3a}$$

or

$$T^* = q_0 x/k \quad \text{for UHF.} \tag{3b}$$

In terms of the dimensionless number Π for laminar natural convection, the similarity variable, the dimensionless stream function and the dimensionless temperature can be respectively defined as

$$\eta = (y/x)\Pi^{1/n} \tag{4}$$

$$f = \psi/a\Pi^{1/n} \tag{5}$$

and

$$\theta = (T - T_\infty)/(T_0 - T_\infty) \quad \text{for UWT} \tag{6a}$$

or

$$\varphi = [(T - T_\infty)/(q_0 x/k)]\Pi^{1/n} \quad \text{for UHF.} \tag{6b}$$

The value of the constant n depends on the case of natural convection. For a vertical plate with UWT, $n = 4$, and for that with UHF, $n = 5$. For horizontal plates with UWT and UHF, $n = 5$ and 6, respectively.

With these dimensionless variables, similarity transformations of the boundary-layer equations can be obtained [6, 8–10]. The local friction coefficient and Nusselt number, for fluids of any Prandtl number between 0.001 and infinity, can be calculated from the following equations:

$$C_f = \tau_0/\rho(\alpha\nu/x^2) = \Pi^{3/n} f''(0) \tag{7}$$

$$Nu = hx/k = \Pi^{1/n}[-\theta'(0)] \quad \text{for UWT} \tag{8a}$$

or

$$Nu = \Pi^{1/n}[1/\varphi(0)] \quad \text{for UHF.} \tag{8b}$$

Dividing equation (7) by equation (8) yields

$$\frac{C_f/2}{Nu\,Ra^{2/n}} = \frac{1}{2}\left(\frac{Pr}{1+Pr}\right)^{2/n}\left[\frac{f''(0)}{-\theta'(0)}\right] \quad \text{for UWT} \tag{9a}$$

or

$$\frac{C_f/2}{Nu\,Ra^{2/n}} = \frac{1}{2}\left(\frac{Pr}{1+Pr}\right)^{2/n}[f''(0)\varphi(0)] \quad \text{for UHF.} \tag{9b}$$

NOMENCLATURE

c	constant
C	concentration
C_f	friction coefficient
C^*	characteristic concentration, $(C_0 - C_\infty)$ for uniform wall concentration case; $J_0 x/D$ for uniform mass flux case
D	molecular diffusivity
f	dimensionless stream function
g	gravitational acceleration
h	local heat transfer coefficient
h_m	local mass transfer coefficient
J	molecular diffusion flux
k	thermal conductivity of fluid
m, n	constants
Nu	local Nusselt number, hx/k
Pr	Prandtl number, ν/α
q	heat flux
Ra	local Rayleigh number for natural convection heat transfer, $g\beta T^* x^3/\alpha\nu$
Ra_m	local Rayleigh number for natural convection mass transfer, $g\gamma C^* x^3/D\nu$
Sc	Schmidt number, ν/D
Sh	local Sherwood number, $h_m x/D$
T	temperature
T^*	characteristic temperature, $(T_0 - T_\infty)$ for UWT case; $q_0 x/k$ for UHF case
x, y	coordinates parallel and normal to the plate.

Greek symbols

α	thermal diffusivity
β	thermal expansion coefficient
γ	concentration expansion coefficient
η	dimensionless coordinate, $(y/x)\Pi^{1/n}$
θ	dimensionless temperature for the UWT case, $(T-T_\infty)/(T_0-T_\infty)$
ν	kinematic viscosity
Π	dimensionless number, equation (1)
ρ	density of fluid
τ	shear stress
φ	dimensionless temperature for the UHF case, $[(T-T_\infty)/(q_0 x/k)]\Pi^{1/n}$.

Subscripts

m	mass transfer
0	adjacent to the wall
∞	far from the wall.

2.2. *Analogy for large Prandtl numbers*

The numerical solutions of $f''(0)$, $\theta'(0)$ and $\varphi(0)$ for $0.001 \leqslant Pr \leqslant 10\,000$ are available from refs. [8–10] for various laminar natural convection cases. The high accuracy of the numerical results have been verified in refs. [8–10] by comparing with reported similarity solutions. From these data the values of $C_f/(2Nu\,Ra^{2/n})$ for some specific Prandtl numbers have been calculated and listed in Table 1. It is very interesting to find that, for each case of laminar natural convection, the values of $C_f/(2Nu\,Ra^{2/n})$ for Prandtl numbers greater than 0.7 are fairly close to a constant, as can be seen from the lower part of Table 1. Therefore, the analogy of fluid friction and heat transfer for large Prandtl numbers can be proposed as

$$\frac{C_f/2}{Nu\,Ra^{2/n}} = c. \tag{10}$$

Similarly, the analogy of wall friction and mass transfer for large Schmidt numbers can be expressed as

$$\frac{C_f/2}{Sh\,Ra_m^{2/n}} = c \tag{11}$$

where the local Rayleigh number Ra_m for natural convection mass transfer is defined as

$$Ra_m = g\gamma C^* x^3/D\nu \tag{12}$$

with the characteristic concentration defined by

$$C^* = C_0 - C_\infty \text{ for the uniform wall concentration case} \tag{13a}$$

or

$$C^* = J_0 x/D \text{ for the uniform mass flux case.} \tag{13b}$$

Table 1. Values of $C_f/(2Nu\,Ra^{2/n})$

	Vertical plate		Horizontal plate	
Pr	UWT ($n=4$)	UHF ($n=5$)	UWT ($n=5$)	UHF ($n=6$)
0.001	1.24	2.29	2.07	2.88
0.01	1.23	1.84	1.63	1.99
0.1	1.18	1.49	1.30	1.43
0.7	1.14	1.31	1.13	1.17
1	1.13	1.29	1.11	1.14
10	1.13	1.23	1.06	1.07
100	1.15	1.22	1.06	1.05
1000	1.16	1.22	1.07	1.05
10 000	1.16	1.22	1.07	1.05

2.3. *Analogies for any Prandtl number between* 0.001 *and infinity*

By multiplying a factor of $[Pr/(1+Pr)]^{1/m}$ to the relation (10), the analogy of friction and heat transfer for fluids of any Prandtl number from 0.001 to infinity can be obtained for each case of laminar natural convection. The proposed analogies and the proportional constants (presented in Table 2) are summarized below.

(1) Vertical plate with uniform wall temperature or concentration

$$\frac{C_f/2}{Nu\,Ra^{1/2}}\left(\frac{Pr}{1+Pr}\right)^{1/100} = \frac{C_f/2}{Sh\,Ra_m^{1/2}}\left(\frac{Sc}{1+Sc}\right)^{1/100} = 1.15. \tag{14}$$

(2) Vertical plate with uniform flux

$$\frac{C_f/2}{Nu\,Ra^{2/5}}\left(\frac{Pr}{1+Pr}\right)^{1/11} = \frac{C_f/2}{Sh\,Ra_m^{2/5}}\left(\frac{Sc}{1+Sc}\right)^{1/11} = 1.21. \tag{15}$$

(3) Horizontal plate with uniform wall temperature or concentration

$$\frac{C_f/2}{Nu\,Ra^{2/5}}\left(\frac{Pr}{1+Pr}\right)^{1/10} = \frac{C_f/2}{Sh\,Ra_m^{2/5}}\left(\frac{Sc}{1+Sc}\right)^{1/10} = 1.05. \tag{16}$$

Table 2. Values of $(C_f/(2Nu\,Ra^{2/n}))[Pr/(1+Pr)]^{1/m}$ for various laminar natural convection cases

	Vertical plate		Horizontal plate	
Pr	UWT ($n = 4$, $m = 100$)	UHF ($n = 5$, $m = 11$)	UWT ($n = 5$, $m = 10$)	UHF ($n = 6$, $m = 7$)
0.001	1.16	1.22	1.04	1.07
0.01	1.17	1.21	1.03	1.03
0.1	1.15	1.20	1.02	1.01
0.7	1.13	1.21	1.03	1.03
1	1.13	1.21	1.03	1.04
10	1.13	1.22	1.05	1.05
100	1.15	1.22	1.06	1.05
1000	1.16	1.22	1.07	1.05
10 000	1.16	1.22	1.07	1.05

(4) Horizontal plate with uniform flux

$$\frac{C_f/2}{Nu\,Ra^{1/3}}\left(\frac{Pr}{1+Pr}\right)^{1/7} = \frac{C_f/2}{Sh\,Ra_m^{1/3}}\left(\frac{Sc}{1+Sc}\right)^{1/7} = 1.04. \quad (17)$$

3. CONCLUSIONS

The analogies between fluid friction and heat transfer of laminar natural convection on flat plates have been proposed as $C_f/(2Nu\,Ra^{2/n}) = c$ for large Prandtl numbers ($Pr \geqslant 0.7$) and $(C_f/(2Nu\,Ra^{2/n}))[Pr/(1+Pr)]^{1/m} = c$ for any Prandtl number between 0.001 and infinity. The values of n and m, as well as the proportional constant c, have been determined for various cases of laminar natural convection on vertical and horizontal flat plates maintained with uniform wall temperature or heat flux. These analogies are applicable to any incompressible Newtonian fluid with constant physical properties except density.

The success of the present analogies may encourage the development of more analogies for other natural convection systems.

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